Original Article

International Journal of Fundamental Research

Volume 1 Issue 1 (June - 2026) ISSN – xxxx-xxxx

MODIFIED EQUATIONS OF ROTATIONAL MOTION OF AN OBLATE SATELLITE IN THE RESTRICTED THREE – BODY PROBLEM

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ARTICLE INFO

Keywords:
Eulerian angles
Oblate spheroid
Libration points
Restricted three-body problem

ABSTRACT

The study revisits the dynamics of an oblate satellite subjected to the gravitational fields of two spherical primaries, a problem traditionally approached via the translational and rotational motion equations of Kondurar and Shinkarik (1972). While the translational equations adequately predict the conditions for libration points, the conventional rotational equations fail to provide a general framework for determining these equilibrium configurations. Motivated by earlier works (Choudhry, 1977; Kumar & Choudhry, 1986; Ahmad and Singh, 1995), this paper derives modified equations of rotational motion for an oblate satellite. By considering the satellite as an oblate spheroid with distinct moments of inertia about its equatorial and polar diameters, and employing Eulerian angles alongside a coordinate transformation that aligns the system with the primaries' circular orbits, we reformulate the rotational dynamics. The derivation involves a detailed analysis of the potential function and kinetic energy, leading to a set of modified Lagrange equations that successfully generalize the conditions for libration points under rotational motion. The resultant equations not only address the limitations of the classical formulation but also provide deeper insights into the equilibrium configurations of satellites in multi-body gravitational fields. These advancements hold significant implications for celestial mechanics and the precise design of satellite orbits in complex gravitational environments.

1. INTRODUCTION

In stellar system numerous examples are available where an oblate body is moving under the gravitational field of two spherical bodies. Choudhry (1977), Kumar & Choudhry (1986) and Ahmad and Singh 1995 studied similar problems for the existence of the libration points in the generalized Restricted three-body problem by generalizing the equations of motion of the satellite given by Kondurar and Shinkarik in 1972.

The equations of translational motion of a satellite given by Kondurar and Shinkarik are:

$$\left.\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} \\ \ddot{z} &= \frac{\partial \Omega}{\partial z} \end{aligned}\right\} \quad (1)$$

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View Online: https://pub.aietr.ac.in/IJFR/view-article?pii=S0000000026582272
Received 02 Feb 2026; Revised 04 Mar 2026; Accepted 09 Apr 2026; Available online 12 Apr 2026
Published 15 June 2026.
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where

$$\Omega = \frac{1}{2}(x^2 + y^2) + U \tag{2}$$

and U is the potential function of the satellite due to the two primaries and dot denotes the differentiation with respect to time t.

The equations of rotational motion of oblate satellite given by Kondurar & Shinkarik (1972) in terms of Eulerian angles, are:

$$\ddot{\psi} = -2(\dot{\psi} + 1)\,\dot{\theta}\cot\theta + \varepsilon\dot{\theta}\csc\theta + \frac{\csc^2\theta}{A}\frac{\partial\Omega}{\partial\psi} \tag{3}$$

$$\ddot{\theta} = (\psi + 1)^2 \sin\theta\cos\theta - \varepsilon(\psi + 1)\sin\theta\,\frac{1}{A}\,\frac{\partial\Omega}{\partial\psi} \tag{4}$$

where m_A is the principal moment of inertia of the satellite of mass m.

The conditions for the existence of the libration points are:

$$\frac{\partial\Omega}{\partial x} = \frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial z} = 0 \text{ (for translational motion)} \tag{5}$$

$$\text{and,} \quad \frac{\partial\Omega}{\partial\theta} = \frac{\partial\Omega}{\partial\psi} = 0 \text{ (for rotational motion)} \tag{6}$$

But at the libration points, we know that

$$\dot{x} = \dot{y} = \dot{z} = 0 = \ddot{x} = \ddot{y} = \ddot{z} \tag{7}$$

as well as

$$\dot{\theta} = \dot{\varphi} = \dot{\psi} = 0 = \ddot{\theta} = \ddot{\psi} \tag{8}$$

Combining (1) & (7) we get the conditions (5) for the existence of the libration points, i.e., the equations of translational motion of the satellite are errorless.

But the combination of (3), (4) & (8) yields

$$\frac{\partial\Omega}{\partial\Psi} = 0$$

and

$$\frac{\partial\Omega}{\partial\theta} = mA\,\sin\theta\,(\varepsilon - \cos\theta) = 0 \text{ for } \theta = n$$

$$\neq 0 \text{ for any arbitrary value of } \theta.$$

Therefore, the equations (3) & (4) are not general and hence some modification is required to generate the conditions of libration points is general.

Here our main purpose is to deduce the generalized equations of rotational motion of the oblate satellite in the gravitational field of two primaries (point masses).

Thus, the equations (3) & (4) given by Kondurar & Shinkarik is particular case for generating the conditions for libration point under rotational motion.

2. DERIVATION OF THE MODIFIED EQUATIONS OF ROTATIONAL MOTION OF AN OBLATE SATELLITE:

Let two primaries of masses m_1 and m_2 $(m_1 \geq m_2)$ be located at A_1 and A_2 respectively and let they rotate in circular orbits with constant angular velocity n, keeping the distance A_1A_2 constant, about their centre of mass O which is assumed to be at rest. Let (ξ_i, η_i, ζ_i); $i = 1, 2$ be the coordinate of the primaries A_i referred to the frame $(O, \xi\eta\zeta)$. Here limits of mass, length and time are so chosen that $m_1 + m_2 = 1, A_1A_2 = 1$ and $n = 1$, then as it is known, the gravitational constant has the value unity. We consider the infinitesimal mass (satellite) as an oblate spheroid with centre at $P\,(\xi, \eta, \zeta)$ referred to $(O, \xi\eta\zeta)$ in the gravitational field of the primaries of masses m_1 & m_2, which do not influence the circular motions of the primaries. If $PA_i = r_i; (i = 1,2)$ then the potential of the oblate spheroid (satellite) due to the two masses m_1 and m_2, is given by

$$U = \sum_{i=1}^{2} m\left[\frac{m_i}{r_i} + \frac{(2\mathrm{A} + C - 3\mathrm{I}_i)}{2\mathrm{r}_i^3}\right] \tag{9}$$

where m_A and m_C are the moments of inertia of the satellite about its equatorial and polar diameters respectively and m_{I_i} is the moment of inertia of the oblate satellite relative to the straight line PA_i.

It λ_i, μ_i, ν_i are the direction cosines of PA_i relative to the principal axes of the oblate satellite, then

$$I_i = A(\lambda_i^2 + \mu_i^2) + C\nu_i^2; \lambda_i^2 + \mu_i^2 + \nu_i^2 = 1 \tag{10}$$

From (9) & (10)

$$U = \sum_{i=1}^{2} m\left[\frac{m_i}{r_i} + \frac{I}{r_i^2}(1 - 3\nu_i^2)\right] \tag{11}$$

where

$$\left.\begin{aligned} r_i^2 &= (\xi-\xi_i)^2+(\eta-\eta_i)^2+(\zeta-\zeta_i)^2, I=\frac{C-A}{2} \\ v_i &= (\xi-\xi_i)a_{13}+(\eta-\eta_i)a_{23}+(\zeta-\zeta_i)a_{33} \end{aligned}\right\} \quad (12)$$

[From (1), (2), (3)]

where a_{13}, a_{23}, a_{33} are the direction cosines of the polar diameter of the satellite (Duboshin, 1964).

These direction cosines are expressible in terms of Eulerian angles as

$$a_{13} = \sin\psi\,.\sin\theta\,, a_{23} = -\cos\psi\,.\cos\theta\,, a_{23} = \cos\theta \quad (13)$$

where

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \quad (14)$$

Now we take, as usual, the line A_1A_2 as the x −axis, a line through O and perpendicular to the x −axis as the y −axis and the axis of rotation of the line A_1A_2 as the z −axis. If (x, y, z) be the new coordinates of $P\ (\xi, \eta, \zeta)$ then at any time t, the transformations are given by

$$\left.\begin{aligned} x &= \xi\cos t - \eta\sin t \\ y &= \xi\sin t + \eta\cos t \\ z &= \zeta \end{aligned}\right\} \quad (15)$$

If p, q, r denotes the projections of the angular velocity of rotation of the satellite on the principal axes of the satellite, then

$$\left.\begin{aligned} p &= \dot\psi\sin\theta\,.\sin\phi + \dot\theta\cos\phi \\ q &= \dot\psi\sin\theta\,.\cos\phi + \dot\theta\sin\phi \\ r &= \dot\psi\cos\theta + \dot\phi \end{aligned}\right\} \quad (16)$$

Now the K.E. of the satellite $P\ (x, y, z)$ is given by

$$T = \frac{m}{2}[\dot x^2+\dot y^2+\dot z^2+2(x\dot y-\dot x y)+x^2+y^2] + \frac{m}{2}[A(p^2+q^2)+Cr^2]$$

$$i.e.\ T = \frac{m}{2}[\dot x^2+\dot y^2+\dot z^2+2(x\dot y-\dot x y)+x^2+y^2] + \frac{m}{2}\left(\dot\psi^2\sin^2\theta+\dot\theta^2\right) + C(\dot\psi^2\cos^2\theta + \dot\phi^2+2\dot\psi\dot\phi\cos\theta)] \quad (17)$$

From (17) it is clear that T is independent of z and consequently T does not depend upon the angle of precession ψ.

Also, from (10), (11), (12) & (13) it is clear that U is independent of ϕ and thus the Lagranges equations of translational motion of the satellite $P\ (x, y, z)$ are

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial\dot x}\right)-\frac{\partial T}{\partial x} &= \frac{\partial U}{\partial x} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot y}\right)-\frac{\partial T}{\partial y} &= \frac{\partial U}{\partial y} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot z}\right) &= \frac{\partial U}{\partial z} \end{aligned}\right\} \quad (18)$$

and the Lagrange's equations of rotational motion of the satellite $P\ (x, y, z)$ are

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right) &= \frac{\partial U}{\partial\psi} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)-\frac{\partial T}{\partial\theta} &= \frac{\partial U}{\partial\theta} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right)-\frac{\partial T}{\partial\phi} &= 0 \end{aligned}\right\} \quad (19)$$

Let us take

$$\mu = \frac{m_2}{m_1+m_2} = m_2$$

as

$$\mathrm{m}_1 + m_2 = 1$$

then

$$\mathrm{m}_1 = 1-\mu.$$

Let $(x_i, 0, 0);\ (i = 1,2)$ be the new coordinates of the primaries A_i relative to the frame (O, xyz), then

where

$$\left.\begin{aligned} r_i^2 &= (x-x_i)^2+y^2+z^2 \\ x_1 &= -\mu, x_2 = 1-\mu \end{aligned}\right\} \quad (20)$$

Now differentiating p, q, r partially with respect to $\dot\phi$ and ϕ we get

$$\left.\begin{aligned} \frac{\partial p}{\partial\dot\phi} &= \frac{\partial q}{\partial\dot\phi} = 0\,, \frac{\partial r}{\partial\dot\phi} = 1 \\ \frac{\partial p}{d\phi} &= \dot\psi\sin\theta\cos\phi - \dot\theta\sin\phi \\ \frac{\partial q}{\partial\phi} &= -\dot\psi\sin\theta\sin\phi - \dot\theta\cos\phi \end{aligned}\right\} \quad (21)$$

$$\frac{\partial r}{\partial\phi} = 0$$

Here from (17) and (21)

$$\left.\begin{aligned}\frac{\partial T}{\partial \dot{\phi}} &= mC[\dot{\phi} + \dot{\psi}\cos\theta] \\ \frac{\partial T}{\partial \phi} &= 0\end{aligned}\right\} \qquad (22)$$

The combination of third equation of (19) & (22) yield

$$mC\frac{d}{dt}\,(\dot{\phi} + \dot{\psi}\cos\theta\,) = 0$$

Integrating with respect to t, we get

$$mC\big(\dot{\psi}\cos\theta + \dot{\phi}\big) = constant = \varepsilon mA.$$

Dividing by m, we get

$$\text{i.e. } C\big(\dot{\psi}\cos\theta + \dot{\phi}\big) = \varepsilon A \qquad (23)$$

where A and C are the moments of inertia of the satellite about its equatorial and polar diameters, respectively. This is consistent with equation (17), where the rotational kinetic-energy term contains A and C.

Differentiating (17) partially with respect to ψ and introducing (23) in it we get

$$\frac{\partial T}{\partial \psi} = A\left(\dot{\psi}\sin^2\theta + \varepsilon\cos\theta\right) \qquad (24)$$

Introduction of (24) in the first equation of (19) yields

$$\frac{d}{dt}\big[A\big(\dot{\psi}\sin^2\theta + \varepsilon\cos\theta\big)\big] = m\frac{\partial \Omega}{\partial \psi}$$

$$\Rightarrow mA\big(\ddot{\psi}\sin^2\theta + 2\dot{\psi}\dot{\theta}\sin\theta\cos\theta - \varepsilon\theta\sin\theta\big) = m\frac{\partial \psi}{\partial \Omega}$$

$$\Rightarrow \ddot{\psi} = -2\dot{\psi}\dot{\theta}\cot\theta + \varepsilon\dot{\theta}\csc\theta + \frac{\csc^2\theta}{A}\,\frac{\partial \Omega}{\partial \psi}$$

Similarly differentiating (17) partially with respect to $\dot{\theta}$ and θ, we get

$$\frac{\partial T}{\partial \dot{\theta}} = \bar{A}\dot{\theta} \qquad (25)$$

$$and\ \frac{\partial T}{\partial \theta} = \bar{A}\big[\dot{\psi}^2\,sin\,\theta\,cos\,\theta - \varepsilon\dot{\psi}\,sin\,\theta\big] \qquad (26)$$

Introduction of (25) and (26) in the second equation of (19) yields

$$\frac{d}{dt}\big(\bar{A}\dot{\theta}\big) - \bar{A}\big[\dot{\psi}^2\sin\theta\cos\theta - \varepsilon\dot{\psi}\sin\theta\big] = \frac{\partial U}{\partial \theta}$$

$$\Rightarrow \bar{A}\big[\ddot{\theta} - \dot{\psi}^2\sin\theta\cos\theta + \varepsilon\dot{\psi}\sin\theta\big] = m\frac{\partial \Omega}{\partial \theta}$$

$$\Rightarrow \ddot{\theta} = \dot{\psi}^2\sin\theta\cos\theta - \varepsilon\dot{\psi}\sin\theta + \frac{1}{A}\frac{\partial \Omega}{\partial \theta}$$

Thus, the modified equations of rotational motion of oblate satellite are:

$$\ddot{\psi} = -2\dot{\psi}\dot{\theta}\cot\theta + \varepsilon\dot{\theta}\csc\theta + \frac{\csc^2\theta}{A}\frac{\partial \Omega}{\partial \psi} \qquad (27)$$

and

$$\ddot{\theta} = \dot{\psi}^2\sin\theta\cos\theta - \varepsilon\dot{\psi}\sin\theta + \frac{1}{A}\frac{\partial \Omega}{\partial \theta} \qquad (28)$$

which are different from the equations given by Kondurar & Shinkarik (1972) as in (3) & (4).

3. JUSTIFICATION:

At the libration points $\dot{\psi} = \dot{\theta} = \dot{\phi} = 0 = \ddot{\psi} = \ddot{\theta}$ produce $\varepsilon = 0$ and hence we find

$$\frac{\partial \Omega}{\partial \psi} = \frac{\partial \Omega}{\partial \theta} = 0$$

which are the required conditions for the libration points under rotational motion of an oblate satellite.

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